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The geometry of the Hessian locus: from a theorem of Gordan and Noether to cubic fourfolds, via Gorenstein rings

Abstract:

In algebraic geometry a central role is often played by projective hypersurfaces. A lot of mathematicians have studied their geometric and algebraic properties, also reflected in the so called hessian locus, which we will analyze in this thesis. Interest in this natural construction associated to a projective hypersurface goes back centuries.

If $X = V(f) \subset \mathbb{P}^n$ is a hypersurface, not necessarily smooth, defined by a homogeneous polynomial $f \in \mathbb{K}[x_0, \cdots, x_n]_d$ of degree d (where $\mathbb{K}$ is an algebraically closed field of characteristic 0), we can then naturally define the Hessian matrix of f as the square symmetric matrix, whose entries are the second partial derivatives of f with respect to the x_i's, i.e. the matrix

$$\mathrm{Hess}(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=0,\cdots,n}.$$

The *Hessian locus* $\mathcal{H}_f$ of X is then defined as the zero locus $V(\mathrm{hess}(f))$, where $\mathrm{hess}(f)$ is the determinant of $\mathrm{Hess}(f)$, which is either a homogeneous polynomial of degree $(d-2)(n+1)$ (this happens, for example, if X is a non-singular hypersurface) or is identically zero. The problem of characterizing the hypersurfaces $X = V(f)$ for which the determinant $\mathrm{hess}(f)$ is identically zero has a long history. Since the middle of the 19^{th} century, several authors have worked on this problem: in the 1850's, both in [Hes51] and [Hes59], Hesse proposed a remarkable equivalence, by claiming that a hypersurface defined by a polynomial with vanishing hessian has necessarily to be a cone (the converse is clearly true). This happened to be false and in a fundamental paper of 1876 ([GN76]) Gordan and Noether proved the following:

Theorem A. ***(Gordan-Noether)***
Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface defined over a field $\mathbb{K}$ of characteristic 0 and assume that $\mathrm{hess}(f) \equiv 0$. Then, if $n \leq 3$, X is a cone.

They introduced the fundamental restriction on the admissible dimension of the projective space and provided counterexamples with $n \geq 4$. The so called "Perazzo cubic 3-fold" in $\mathbb{P}^4$ (introduced in [Per00]) is the simplest such counterexample, which will be analyzed in this thesis (Section 1.5).
In the first chapter of this thesis, we will give a new proof of this fundamental result due to Gordan and Noether. We will actually show a theorem (see the following Theorem B), equivalent to Gordan and Noether's result, which deals with the so called *standard Artinian Gorenstein algebras.* A standard Artinian Gorenstein $\mathbb{K}-$algebra (in the thesis abbreviated to SAGA) is an Artinian graded $\mathbb{K}-$algebra $R = \oplus_{i=0}^{N} R^i$, with the R^i's of finite dimension, such that R is standard (i.e. generated in degree 1) and it satisfies the Poincaré-Gorenstein duality (i.e. $R^N \simeq R^0 \simeq \mathbb{K}$ and the pairing given by the multiplication map $R^i \times R^{N-i} \to R^N$ is perfect). An example is given by the even cohomology ring of an oriented compact Kähler variety X of even dimension which is generated in degree 2. In this setting the well-known *Hard Lefschetz Theorem* holds and a natural question is whether a general standard Artinian Gorenstein algebra satisfies an analogous property. In the 80's, inspired by the aforementioned theorem, the so-called Lefschetz properties for an Artinian algebra were defined. Roughly speaking, we say that an algebra R satisfies the *weak Lefschetz property* if the multiplication map $x\cdot : R^k \to R^{k+1}$ is of maximal rank for all $k \geq 0$ and $x \in R^1$ general. In a similar way, the *strong Lefschetz property* is defined, when the same property holds for multiplication maps by powers of general elements in degree 1. In the first chapter of this thesis we prove the following

Theorem B.
For all standard Artinian Gorenstein $\mathbb{K}-$algebras R, such that $\dim(R^1)$ is at most 4, the strong Lefschetz property in degree 1 holds, i.e. there exists an element $x \in R^1$ such that the multiplication map $x^{N-2}\cdot : R^1 \to R^{N-1}$ is an isomorphism.

Despite of the algebraic nature of the statement, our proof of Theorem B is characterized by a geometric approach and gives, as a byproduct, a new proof of Theorem A due to Gordan and Noether, which Theorem B is equivalent to, as said above.

The interesting and, in some sense, surprising equivalence between Theorem A and Theorem B (see Section 1.4 or [HMM+13, Rus16] for example) is realized by a connection between these different settings based on Macaulay's theory of Inverse Systems ([HMM+13, Theorem 2.71] or the original [Mac94]), which allows to construct any standard Artinian Gorenstein algebra, from a homogeneous form in a finite number of variables.

A natural question arising from our work is whether the methods used to prove Theorem B have more applications and, in particular, if they could be applied to study problems related to other strong or weak Lefschetz properties for Gorenstein rings, which, let us stress, are known for only few Artinian algebras. With this in mind, in Chapter 2, we treat some open cases, by focusing on a particular example of standard Artinian Gorenstein $\mathbb{K}-$ algebras, namely the Jacobian ring of a smooth hypersurface. In general, if $X = V(f) \subset \mathbb{P}^n$ is a smooth hypersurface of degree d, with $f \in \mathbb{K}[x_0, \cdots, x_n]_d$, one can consider the Jacobian ideal of f

$$J_f = \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

generated by the partial derivatives of f with respect to the x_i's and the Jacobian ring $R = S/J_f$. In particular, we deal with Jacobian rings of smooth cubic threefolds in $\mathbb{P}^4$ and smooth cubic fourfolds in $\mathbb{P}^5$ and we prove the following:

Theorem C.
Let R be the Jacobian ring of a smooth cubic threefold. Then R satisfies the strong Lefschetz property, i.e.

the general element $x \in R^1$ is such that the multiplication maps $x^3 \cdot : R^1 \to R^4$ and $x \cdot : R^2 \to R^3$ are both isomorphisms.
If R is the Jacobian ring of a smooth cubic fourfold, then for a general $x \in R^1$ the multiplication map $x^4 \cdot : R^1 \to R^5$ is a bijection, i.e. the strong Lefschetz property holds in degree 1.

Theorem C will follow from a more general statement for complete intersection Gorenstein algebras presented by quadrics, i.e. quotients of $\mathbb{K}[x_0, \cdots, x_n]$ by ideals generated by a regular sequence of homogeneous polynomials of degree 2. These results provide evidence for a well known conjecture which states that complete intersection Gorenstein algebras in characteristic 0 should satisfy the Lefschetz properties (see for example [HMM+13, Conjecture 3.46]). Furthermore, we will extend our proof of some of the strong Lefschetz properties to complete intersection Gorenstein algebras presented by quadrics, when the dimension of R^1 is larger.

In the third chapter, we continue the study of cubic hypersurfaces from the perspective of their Hessian loci. As observed above, given a smooth cubic hypersurface $X = V(f) \subset \mathbb{P}^n$, we can consider the associated Hessian locus $\mathcal{H}_f$, which is a hypersurface of degree $n+1$. By studying the nature of the Hessian matrix and its connections with elements in the Jacobian ideal of f, interesting phenomena arise. By exploiting properties of this type, as also a stratification of $\mathbb{P}^n$ given by the decreasing rank of the Hessian matrix via the evaluation map, we will analyze the geometry of these Hessian hypersurfaces, of their singular loci and their desingularizations.
The geometry of cubic hypersurfaces in $\mathbb{P}^n$ and their Hessians has been studied by many authors (see for example [CO20, GR15, Huy]). In particular, for $n = 3$, in [DvG07] the authors study the classical case of the general cubic surface and of the associated Hessian quartic surface, which is singular in exactly 10 isolated points. Moreover, [AR96, Appendix IV] studies the case of cubic threefold in $\mathbb{P}^4$. In particular, the author considers the Hessian quintic threefold $\mathcal{H}$ associated to a general cubic threefold and constructs a correspondence variety over $\mathcal{H}$, which represents a desingularization of the Hessian hypersurface. Adler, not only shows that in the general case this Hessian hypersurface is singular along a curve, but he also studies the geometric properties of such a curve, such as smoothness and irreducibility, and computes its degree and its genus.

With this purpose, we start analyzing and generalizing in every dimension some constructions described in [AR96, Appendix IV]. By studying the rank of the Hessian matrix, we will consider the loci

$$\mathcal{D}_k(f) = \{[x] \in \mathbb{P}^n \mid \operatorname{Rank}(\operatorname{Hess}(f)|_x) \leq k\},$$

which will be identified with the intersections $\mathcal{Q}_k \cap \mathbb{P}(J_f^2)$, where $\mathcal{Q}_k$ is the locus of quadrics in $\mathbb{P}^n$ whose rank is at most k. By using the results described for example in [Har95], we can then get the expected dimension and the degree (if they are non-empty) of the loci $\mathcal{D}_k$'s. This can be seen as a first step in the analysis of the Hessian hypersurface $\mathcal{H}$, since the loci $\mathcal{D}_k$'s are strongly related with the singularities of $\mathcal{H}$, indeed, by generalizing one of the results presented in [AR96, Appendix IV], we prove the following:

Theorem E.
For any smooth cubic hypersurfaces $V(f) \subset \mathbb{P}^n$, we have that $\mathcal{D}_{n-1}(f) = \operatorname{Sing}(\mathcal{H})$.

To prove the above result, we will consider a correspondence variety over $\mathcal{H}$ which can be described as

$$\Gamma_f = \{([v], [w]) \in \mathbb{P}^n \times \mathbb{P}^n \mid \partial_v \partial_w(f) = 0\},$$

and which has also an important intrinsic geometric meaning: indeed, by also exploiting the existence of an involution, naturally defined over Γ_f, we show the following

Theorem F.
For the general smooth cubic hypersurface $V(f)$, we have that Γ_f is smooth and the natural projection $\pi_1 : \Gamma_f \to \mathcal{H}$ is a desingularization for the Hessian hypersurface.

We also show that for a general smooth cubic fourfold $X = V(f)$ another property of the loci $\mathcal{Q}_k$ is inherited by the loci $\mathcal{D}_k(f)$: indeed we have that $\mathrm{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$. This means that for f general, the locus $\mathcal{D}_k(f)$ is smooth outside the points where the Hessian matrix has rank strictly smaller than k. Since in [RV17] the authors show that for $V(f)$ smooth and general cubic fourfold in $\mathbb{P}^5$, the locus $\mathcal{D}_3(f)$ is empty, we get that in the general case the Hessian hypersurface associated to a smooth cubic fourfold is singular over a surface which is smooth. On the other hand, looking at the expected dimensions of these loci, one has that for bigger dimensions (namely for Hessians associated to cubic hypersurfaces in $\mathbb{P}^n$, with $n \geq 6$) the singular locus of the Hessian hypersurface is itself singular. It is then natural to approach the study in the case of $\mathbb{P}^5$, the first open and the last one with a smooth singular locus, for f general. In this last part, we will set the field $\mathbb{K} = \mathbb{C}$, since we will use also techniques and notions of singular cohomology, even if some results still hold in a more general setting.
In general, given a vector bundle E and a line bundle L over a projective variety X, one can consider a symmetric vector bundle morphism $\varphi : E \to E^* \otimes L$ and define the loci

$$\mathcal{D}'_k(\varphi) = \{x \in X \mid \mathrm{Rank}(\varphi_x) \leq k\},$$

known as *degeneraci loci*. By considering the symmetric vector bundle map $\varphi = \mathrm{Hess}(f)\cdot : \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}^{n+1}(1)$, given by the multiplication by the Hessian matrix of a cubic hypersurface $V(f)$, it then follows that the loci $\mathcal{D}_k(f)$ considered above coincide with the degeneraci loci $\mathcal{D}'_k(\mathrm{Hess}(f)\cdot)$. By analyzing these loci from this perspective, by using for example results of [FL83, HT90, Tu89], one can show the non-emptiness and the connectedness of suitable loci $\mathcal{D}_k$ and, by using the approach presented in [HT84], one can also calculate some Chern classes. From this, we prove:

Theorem G.
Let $V(f)$ be a general smooth cubic fourfold. Then the singular locus of the associated Hessian hypersurface $Z := \mathrm{Sing}(\mathcal{H}_f)$ is a smooth, irreducible and minimal surface of general type with degree 35 with numerical invariants $K_Z^2 = 315$, geometric genus $p_g(Z) = 55$, irregularity $q(Z) = 0$ and (topological) Euler characteristic $e(Z) = 357$. Moreover, its canonical divisor is $K_Z = 3H|_Z + \eta$ (where H is the hyperplane class in $\mathbb{P}^5$ and η is a non-trivial 2-torsion element in $\mathrm{Pic}^0(Z)$) and Z is projectively normal

To conclude, let us stress that for such a surface Z, we will construct a natural unramified double cover, which will turn out also to be connected (as proved in Appendix A, by using tools coming from representation theory).

In the following, I attach also a partial bibliography of the thesis. The topics and the results of this thesis have also been presented in [BF22] and [BFP22] (the first paper has already been published, the second one has been accepted for publication by *Selecta Mathematica*).

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